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Aggregation of Market Risks using Pair-Copulas*

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Abstract

The advent of the Internal Model Approval Process within Solvency II and the desirability of many insurance companies to gain approval has increased the importance of some topics such as risk aggregation in determining overall economic capital level. The most widely used approach for aggregating risks is the variance-covariance matrix approach. Although being a relatively well-known concept that is computationally convenient, linear correlations fail to model every particularity of the dependence pattern between risks. In this paper we apply different pair-copula models for aggregating market risks that represent usually an important part of an insurer risk profile. We then calculate the economic capital needed to withstand unexpected future losses and the associated diversification benefits. The economic capital will be determined by computing both 99.5^{th}VaR and 99.5^{th}ES following the requirements of Solvency II and SST.

1 Introduction

Insurance companies face a multitude of risks that could cause financial losses. Economic capital is the amount of capital that is needed to cover losses at a certain risk tolerance level. It captures a wide range of risks such as financial, insurance and operational risks and expresses all of this as a single number. Recently, within the regulatory framework of Solvency II, EIOPA addressed risk aggregation as one of the main challenges in risk management. Risk aggregation is about studying aggregate risks of a given portfolio. If we consider a set of risks (e.g. returns or losses in a portfolio), denoted by a random vector $X = (X_1, \ldots, X_d)$, the aim is to build an aggregation function of those risks, denoted $g(X)$, where $g\colon \mathbb{R}^d \to \mathbb{R}$ and to measure the risk of the quantity $F(g(X))$, for some risk measure F over a given period.
For the purpose of economic capital allocation, one can employ different approaches to aggregate risks. The most commonly used method is the variance-covariance matrix approach. While being easily implemented and convenient to work with, this approach can lead to a misestimation of the aggregated economic capital. Hence, a copula-based approach is chosen to achieve this

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[1]The European Insurance and Occupational Pensions Authority is a European Union financial regulatory institution that replaced the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS). EIOPA is part of the European System of Financial Supervision consisting of three European Supervisory Authorities and the European Systemic Risk Board. It is an independent advisory body to the European Parliament and the Council of the European Union. EIOPAs core responsibilities are to support the stability of the financial system, transparency of markets and financial products as well as the protection of insurance policyholders, pension scheme members and beneficiaries.

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purpose. However, one major shortcoming of copulas is their use in high dimensions. Indeed, elliptical copulas can be easily extended to higher dimension, but they are unable to model financial asymmetries (see Patton, 2009), while Archimedean copulas are not satisfactory to describe multivariate dependence in dimensions higher than two (Joe, 1996).
In this paper, using the method of pair-copula models introduced first by Joe (1997), we analyze the impact of aggregating risks for the purpose of computing the economic capital through the calculation of two risk measures, the Value-at-Risk (99.5^{th}VaR) - required by Solvency II - and the Expected Shortfall (99^{th}ES) - required by the Swiss Solvency Test (SST). Furthermore, we compare the economic capital resulting from the different copula models with the most widely used approach in the insurance industry: the variance-covariance matrix and the simple summation approach. The two last approaches are introduced in the last section of this paper. The diversification benefits derived from the different methodologies are then discussed.
Papers in the insurance area, where the application deals with risk aggregation are quite few. Most of them compare the copula methodology with linear aggregation approaches, thereby documenting how the latter may underestimate total risk (Schuermann and Rosenberg 2006, Tang and Valdez 2009). Among the other approaches reviewed in the risk aggregation area, the methodology we have chosen has not been applied yet in the insurance area even if this approach is not new. Indeed, pair-copulas are used to achieve our purpose.
Risk aggregation provides necessary information that enables effective group-wide risk management, as well as a wide variety of other key business decisions and business processes. However, the financial crisis that began in 2007 highlighted at least some degree of failure of risk aggregation methods. Many of the firms acknowledge that model risk in this area may be higher than previously recognized. Despite that recognition, there has been surprisingly little movement by most of these firms to reassess or revise risk aggregation practices in significant ways. Actually, in many Solvency II internal models, dependence is modeled with correlation matrix or Gaussian copula. As the dimension is large in general, the correlation matrix or the Gaussian copula remain the easiest tool (it is sometimes replaced by a Student copula). Nonetheless, other alternatives may be provided such as the use of pair-copula models (Joe, 1996; Bedford and Cooke, 2001-2002; Guégan and Maugis, 2010 and references therein). This represents a radically new way of constructing complex multivariate highly dependent models, which parallels classical hierarchical modeling (Green et al., 2003 and Guégan and Hassani, 2010-2011).
A key contribution of our paper is to provide existing analytical techniques around the dependence modeling that are supervisory expectations in the context of Solvency II. First, we assess several types of vine and nested copula models. This is accomplished through the application of canonical maximum likelihood (CML) estimation methodology. We use empirical marginal distributions and calibrate the copula parameters by maximum pseudo-likelihood estimation (MPLE). We also conduct benchmarking of models by comparing two risk measures and the associated diversification benefits across different frameworks for aggregating risks. Indeed, we compare various copula models, as well as other methods for the construction of a joint distribution of losses, such as the variance-covariance matrix approach and the simple summation.The empirical exercise uses financial data collected from Bloomberg to proxy for various market risk types. The empirical analysis focuses on five of the largest market sub-risks available on a daily frequency (Equity, Interest rates, Spread, Foreign Exchange and Implied Volatility). The rationale for concentrating on these risks is motivated in large part by the fact that they usually represent more than 75% of the market risk portfolio and about 60% of the total risk portfolio. While the focus herein is upon the insurance sector, our methodology could just as easily be extended to any other kind of financial institution such as a bank.
This study is part of a literature that performs a comprehensive analysis around how to combine a set of risk factors influencing the total risk of large insurance companies. The five main market

sub-risks are considered from January 1, 1999 to March 31, 2009.. Each data set is filtered by applying AR-*like*GARCH process using skewed elliptic and GED innovations. We then determine the dependence structure between these five proxies applying the pair-copula methodology. Two kinds of dependence prevail: the Student-t copula and the Student C-vine. We eventually provide the aggregated economic capital through 99.5^{th} VaR and 99^{th}ES and determine the corresponding amount of diversification benefits for each approach.
The rest of the paper is structured as follows. Section 2 provides the methodology used to model the dependence structure. Section 3 deals with the modeling and the filtering process of market risks needed to build and calibrate the pair-copula models produced in Section 4 where the numerical results of the estimation are presented. Section 5 provides a comparison of the economic capital and the diversification benefits resulting from the assuming different copula models to the variance-covariance matrix and simple summation approach. We conclude this paper in Section 6 with few remarks on the findings and an outlook for future research on the subject of aggregating risks for the purpose of setting capital requirements.

2 Methodology used to Model Dependency

A common approach for deriving economic capital is to first assess the individual risk components and then consider possible techniques to aggregate these components to derive an overall capital number. Insurance companies differ in their approaches to economic capital risk aggregation, some techniques being more sophisticated than others. The most commonly used methodologies are simple summation, variance-covariance matrix and copula-based approaches. The method we have chosen to calculate the aggregate economic capital is the pair-copula approach (Joe, 1996), which is more flexible than the use of the variance-covariance matrix. While the latter only requires measurement of the sub-risks' economic capital estimates and the correlation between the sub-risks, copula methods of aggregation depend on the whole distribution of the sub-risks. However, the full marginal risk distribution is needed for each risk, rather than the relevant capital number in the case of the variance-covariance matrix approach.
In the simple case of two risks X and Y, a copula $C\colon [0,1]^2 \to [0,1]$ is a joint cumulative distribution function (cdf) of two random variables which are uniformly distributed on [0,1]. A two-dimensional cdf H with continuous margins F_1 and F_2 can be written as $H(x_1, x_2) = C(F_1(x_1), F_2(x_2))$. In other words, a copula is the description of a multivariate dependence structure. It is the link between the one-dimensional marginal distribution and the n-dimensional joint distribution in the general case. This approach can be extended to aggregate more than two marginal distributions into a joint multivariate distribution.
In the sequel, a short reminder about the pair-copula decomposition is given. For more details about the models and its estimation inference, we refer to Joe (1996), Bedford and Cooke (2001, 2002), Kurowicka and Cooke (2006) and Guégan and Maugis (2010).The decomposition of a multivariate distribution in a cascade of pair-copulas was originally proposed by Joe (1996), and later discussed in details by the cited authors. The method of extending two-dimension copulas to higher dimensions is not very flexible and additional assumptions are often needed. In this section we use the results of Joe (1996) and follow his exposition very closely. The basic idea behind the pair-copula construction is to decompose an arbitrary distribution function into simple bivariate building blocks that are two-dimensional copulas and stitch them together appropriately. Besides,

[2]The risks are represented by the daily Euro Stoxx index for Equity risk, the daily Euro 10 year-IR swap rates for interest rates risk, the daily Euro spread figures for spread risk, the daily Euro Implied Volatility rates for Implied volatility risk and the daily USD/EUR Foreign Exchange rates for FX risk

this method is recursive by nature. Consider the random vector $X = (X_1, \ldots, X_n)$ whose joint density $f(x_1, \ldots, x_n)$ can be factorized as hereafter:

$$f(x_1, \ldots, x_n) = f(x_n) \cdot f(x_{n-1}|x_n) \cdot f(x_{n-2}|x_{n-1}, x_n) \cdot \cdots \cdot f(x_1|x_2, \ldots, x_n) \tag{1}$$

From Sklar's theorem we know that there exists a copula C such that:

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{2}$$

Differentiating this formula by applying the chain rule and working with densities, one gets:

$$f(x_1, \ldots, x_n) = c_{12\ldots n}(F_1(x_1), \ldots, F_n(x_n)) \cdot f_1(x_1) \cdot \cdots \cdot f_n(x_n) \tag{3}$$

for some (uniquely identified) n-variate copula density $c_{1\ldots n}(.)$. For the base case in two dimensions we can easily see that the density function $f(x_1, x_2)$ is given by

$$f(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2) \tag{4}$$

This follows immediately by taking partial derivatives with respect to both arguments in $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, where C is the copula associated with F via Sklar's Theorem. Besides, notice that from the above formula the conditional density of X_2 given X_1 can be determined:

$$f_{2|1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_2(x_2) \tag{5}$$

In Section 4, we use the previous formula to represent this conditional distribution function when x and ν are uniform, i.e.$f(x) = f(\nu) = 1, F(x) = x$ and $F(\nu) = \nu$. That is,

$$F(x|\nu, \theta) = \frac{\delta C_{x\nu}(x, \nu, \theta)}{\delta \nu} \tag{6}$$

where the second parameter always corresponds to the conditioning variable and θ denotes the set of parameters for the copula of the joint distribution function of x and ν. For sampling from pair-copula distributions, one needs the inverse of $F(x|\nu)$ with respect to the first variable x. In Table 1 we give some examples of that function that have been used in our case study.

Copulas	$F(u_1\|u_2, \theta)$ function
Bivariate Gaussian	$F(u_1\|u_2, \rho_{12}) = \phi(\frac{\phi^{-1}(u_1) - \rho_{12}\phi^{-1}(u_2)}{\sqrt{1-\rho_{12}^2}})$
Bivariate Student	$F(u_1\|u_2, \rho_{12}, \nu_{12}) = t_{\nu_{12}+1}(\frac{t_{\nu_{12}}^{-1}(u_1) - \rho_{12} \cdot t_{\nu_{12}}^{-1}(u_2)}{\sqrt{\frac{(nu_{12} + (t_{\nu_{12}}^{-1\,2}))(1-\rho_{12}^2)}{\nu_{12}+1}}}$
Bivariate Clayton	$F(u_1\|u_2, \rho_{12}) = u_2^{-\delta_{12}-1}(u_1^{-\delta_{12}} + u_2^{-\delta_{12}} - 1)^{-1-\frac{1}{\delta_{12}}}$
Bivariate Gumbel	$F(u_1\|u_2, \rho_{12}) = C_{12}(u_1, u_2) \cdot \frac{1}{u_2} \cdot (-log(u_2))^{\delta_{12}-1} \cdot$ $((-log(u_1))^{\delta_{12}} + (-log(u_2))^{\delta_{12}})^{\frac{1}{\delta_{12}}-1}$

Table 1: Examples of the $F(u_1|u_2, \theta)$ function for different bivariate copulas.

We present now other ways to compute practically a high dimensional distribution: Fully nested Archimedean Copulas (FNAC) and vines (more particularly, C-vines).

2.1 Nested Copulas

The idea behind FNAC models is simply to add a dimension step by step. For the seek of illustration, take for instance the nodes u_1 and u_2 coupled through copula C_1, node u_3 coupled with $C_1(u_1, u_2)$ through copula C_2, node u_4 coupled with $C_2(u_3, C_1(u_1, u_2))$ through copula C_3, and finally node u_5 coupled with $C_3(u_4, C_2(u_3, C_1(u_1, u_2)))$ through copula C_4. Hence, the copula for the 5-dimensional case requires four bivariate copulas C_1, C_2, C_3 and C_4, with corresponding generators ϕ_1, ϕ_2, ϕ_3 and ϕ_4 :

$$\begin{aligned} C(u_1, \dots, u_5) &= C_4(u_5, C_3(u_4, C_2(u_3, C_1(u_2, u_1)))) \\ &= \phi_4^{-1}(\phi_4(\phi_3^{-1}(\phi_3(\phi_2^{-1}(\phi_2(\phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3))) + \phi_3(u_4))) + \phi_4(u_5))) \end{aligned}$$

In this structure, all margins are themselves Archimedean copulas. It allows for the specification of $d-1$ copulas and corresponding distributional parameters, while the remaining $(d-1)(d-2)/2$ copulas and parameters are implicitly given through the construction. More specifically, the two pairs (u_1, u_3) and (u_2, u_3) both have copula C_2 with dependence parameter θ_2. Moreover, the three pairs (u_1, u_4), (u_2, u_4) and (u_3, u_4) all have copula C_{31} with dependence parameter θ_3. Hence, when adding variable x to the structure, we specify the relationships between x pairs of variables.
The FNAC is a construction where some technical conditions need to be satisfied for being a proper d-dimensional copula. The consequence of these conditions for the FNAC, if all the generators are of the same type, is that the degree of dependence, as expressed by the copula parameter, must decrease with the level of nesting in order for the resulting d-dimensional distribution to be a proper copula. If the generators belong to different families, the parameter restrictions are even stronger. There are only few generators that can be combined and this method often fails in practice. For nested copula models, the parameters may be estimated by maximum likelihood estimation.

2.2 Vines

There exists many ways of building a pair-copula model. This leads to a large number of possible pair-copulas constructions, for instance in the case of five variables there are 240 different constructions. In order to organize them one can use a graphical model denoted as the regular vine introduced by Bedford and Cooke (2001-2002). In the sequel, we are going to focus on one special case of these regular vines: C-vines (Kurowicka and Cooke, 2004). Fitting a canonical vine to our data set is relevant in our case since a particular variable is known to be a key variable that governs interactions in the data set. In such a situation we decide to locate this variable at the root of the canonical vine. In a case of a d-dimensional C-vine model, the pairs at tree 1 are $1, i$, for $i = 2, \dots, d$, and for tree $T(2 \leq T \leq d)$, the conditional pairs are $T, i|1, \dots, T-1$ for $i = T+1, \dots, d$. That is, for the C-vine, conditional copulas are specified for variables T and $i = 1, \dots, T-1$.
In other words, each tree T_j has a unique node that is connected to $n-j$ edges. We have then n possibilities to choose the key variable in the first tree, $n-1$ possibilities in the second tree,... and 3 possibilities in tree $n-2$. Thus, there are $\frac{n!}{2}$ different C-vines. The general expression for the five-dimensional C-vine model is given hereunder:

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) = {} & f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4) \cdot f(x_5) \cdot \\
& c_{12}(F(x_1), F(x_2)) \cdot c_{13}(F(x_1), F(x_3)) \cdot c_{14}(F(x_1), F(x_4)) \cdot c_{15}(F(x_1), F(x_5)) \cdot \\
& c_{23|1}(F(x_2|x_1), F(x_3|x_1)) \cdot c_{24|1}(F(x_2|x_1) \cdot F(x_4|x_1)) \cdot c_{25|1}(F(x_2|x_1), F(x_5|x_1)) \cdot \\
& c_{34|12}(F(x_3|x_1, x_2), F(x_4|x_1, x_2)) \cdot c_{35|12}(F(x_3|x_1, x_2), F(x_5|x_1, x_2)) \cdot \\
& c_{45|123}(F(x_4|x_1, x_2 x_3), F(x_5|x_1, x_2, x_3))
\end{aligned}
$$

2.3 Estimation Methodology

For evaluating the log-likelihoods in pair-copula models, two assumptions are needed. The observations of each variable are independent over time and the variables are uniform in $[0, 1]$. Then, the following steps should be taken when fitting a pair-copula decomposition.

- A specific factorization is defined. We determine which factors are most dependent by using the Kendall's tau or Spearman's rho coefficients. These ones should be coupled. In case of Student-t copulas these can also be the factors where the bivariate Student-t copula estimation reveals the smallest number of degrees of freedom. In case of a C-vine this can be performed in each tree, or one uses the order of covariates in the first tree to determine the order of nodes (which will be used here).
- Pair-copula types are chosen. The so far simplest and most common way to find out which copula type to use is the graphical analysis.
- The copula parameters are estimated via maximum likelihood estimation, see Guégan and Maugis (2010) for more details about the inference procedure. To do so, we first estimate the parameters of the copulas in tree 1 from the original data. Then, we compute the observations (i.e. conditional distribution functions) for tree 2 using the copula parameters from tree 1 and the marginal conditional distributions. Besides, we iterate steps 1 and 2 over all trees. In each tree, we determine which copula types to use.
- We maximize the full log-likelihood function using the parameters obtained from the step-wise procedure as starting values.

3 Dynamics of Market risks

In this section, we summarize the filtering process of the five market risks needed to perform the pair-copula modeling.
Let consider the daily Euro Stoxx index for Equity risk, denoted $(X_1)_t$, the daily Euro 10 year-IR swap rates for interest rates risk, denoted $(X_2)_t$, the daily Euro spread figures for spread risk, denoted $(X_3)_t$, the daily Euro Implied Volatility rates for Implied volatility risk, denoted $(X_4)_t$ and the daily USD/EUR Foreign Exchange rates for FX risk, denoted $(X_5)_t$. The data set starts from January 1, 1999 to March 31, 2009. These samples cover from 10 years and encompass several world crises such as the 9/11 terrorist attack of 2001, the mortgage crisis of 2007 and the huge credit and market turmoil of 2008. This data set have been collected from Bloomberg.
To achieve stationarity, the log returns of the time series, denoted $(R_i)_t$ for $i = 1, \ldots, 5$ are computed: $R_t = ln((X_t)/(X_{t-1}))$. The Augmented Dickey Fuller (ADF) test performed for all

data sets confirm the stationarity of the series. Table 2 summarizes the descriptive statistics of the five market risk returns $(R_1)_t$, $(R_2)_t$, $(R_3)_t$, $(R_4)_t$ and $(R_5)_t$, the ADF test and the normality tests.

3.1 Descriptive Statistics

Statistics	$(R_1)_t$	$(R_2)_t$	$(R_3)_t$	$(R_4)_t$	$(R_5)_t$
Mean	-1.791e-4	-8.061e-5	7.544e-4	2.045e-4	-4.771e-4
Std dev	1.563e-4	9.724e-3	2.278e-2	1.544e-2	6.494e-3
Kurtosis	7.688	8.359	6.987	50.525	4.420
Skewness	-1.328e-2	-4.0463e-2	1.741e-1	2.010	-7.739e-2
ADF test	-1.042	-1.021	-0.835	-0.812	-1.014
p-value	0.01	0.01	0.01	0.01	0.01
JB test	2453.113	3205.662	1787.671	253656.8	228.307
Df	2	2	2	2	2
p-value	$< 2.2e-16$	$< 2.2e-16$	$< 2.2e-16$	$< 2.2e-16$	$< 2.2e-16$
Shapiro test	0.938	0.956	0.936	0.652	0.987
p-value	$< 2.2e-16$	$< 2.2e-16$	$< 2.2e-16$	$< 2.2e-16$	9.607e-15

Table 2: Descriptive statistics of log-returns of $(R_1)_t$, $(R_2)_t$, $(R_3)_t$, $(R_4)_t$ and $(R_5)_t$.

From these findings, all data sets appear to have heavy tails and the normality assumption is inconsistent with all time series. When performing a model selection in the scope of the generalized hyperbolic distribution class based on the Akaike information criterion, we obtain as the best model the symmetric variance gamma distribution (defined as the normal variance-mean mixture where the mixing density is the gamma distribution) for Equity, IR, Implied Volatility and FX risk data while the best model obtained for Spread risk is a symmetric generalized hyperbolic distribution function. The estimated parameters and the Kolmogorov-Smirnov statistics, given in the appendix, confirm theses findings. Hence, as normality assumption is clearly not consistent with all time series, we use Quasi Maximum Likelihood (QML) for estimating the parameters of the AR-*like*GARCH models in the following section.

3.2 Modeling and Filtering Process

When plotting the trajectory for the series $(R_1)_t$,$(R_2)_t$,$(R_3)_t$,$(R_4)_t$ and $(R_5)_t$on the full period from 1999 to 2009 (sample size n=2673), the plots show that the log-returns were more volatile toward the end of the study period. These results might be triggered by the instability in the markets due to the market turmoil starting from 2007. One can expect that the conditional variance of all these series vary over time and hence use GARCH models. Some disadvantage of GARCH processes in modeling financial returns is the symmetry of conditional variance of ϵ_t with respect to positive and negative values of ϵ_{t-1}, ϵ_{t-2},. . . . In practice, one observes a leverage effect, meaning asymmetric consequences of positive and negative innovations (the conditional variance tends to decrease if noise is positive implying bigger returns). An Exponential GARCH process (Nelson, 1991) does not have this disadvantage. This is also the case for Asymmetric

[3]Graphs are provided in the appendix at the end of the paper.

Power ARCH process introduced by Ding, Granger and Engle (1999).
Therefore, in order to filter our market risk data, we apply then AR(k)-*like*GARCH(p,q) process:

$$R_t = \phi_1 R_{t-1} + \cdots + \phi_k R_{t-k} + \epsilon_t, t = 1, \ldots, T. \quad (7)$$

and

$$\epsilon_t = \sigma_t \nu_t \quad (8)$$

where (ν_t) is a strict white noise, and (σ_t) satisfies the following recurrence equations:

- GARCH(p,q) (Bollerslev, 1986)

$$\sigma_t^2 = \omega + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \quad (9)$$

  where ω denotes the variance intercept, α_j the ARCH(p) parameters, β_i the GARCH(q) parameters and σ^2 the conditional variance,

- EGARCH(p,q) (Nelson, 1991)

$$log_e(\sigma_t^2) = \omega + \sum_{j=1}^{p} (\alpha_j z_{t-j} + \gamma_j(|z_{t-j}| - E|z_{t-j}|)) + \sum_{j=1}^{q} \beta_j log_e(\sigma_{t-j}^2) \quad (10)$$

  where the coefficients α_j captures the sign effect, γ_j the size effect and z_t are standardized innovations.

- APARCH(p,q) (Ding, Granger and Engle, 1993)

$$\sigma_t^2 = \omega + \sum_{j=1}^{p} \alpha_j(|\epsilon_{t-j}| - \gamma_j \epsilon_{t-j})^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta \quad (11)$$

  where the power parameter $\delta \in \mathbb{R}^+$, being a Box-Cox transformation of σ_t, and γ_j the leverage(p) parameters. Various models arise from this model such as GARCH (Bollerslev, 1986), AVGARCH (Taylor, 1986 and Schwert, 1990), GJRGARCH (Glosten et al., 1993), TGARCH (Zakoian, 1994), Non linear ARCH (Higgins and Bera, 1992) and the Log ARCH (Geweke, 1986 and Pantula, 1986).

Thus, $Var(\epsilon_t|\epsilon_{t-1}, \epsilon_{t-2}, \ldots) = \sigma_t^2$, $E(\epsilon_t) = 0$, $Cov(\epsilon_t, \epsilon_s) = 0, t \neq s$. The distribution of the residuals ν_t can be Gaussian, Student, GED or the generalized hyperbolic (GH) distributions (Barndorff-Nielsen,1977).
Different AR models are fitted to our time series by applying QML estimation and AIC is used for selecting the number of AR parameters. it comes out that an AR(3) process, an AR(4) process and an AR(6) process can describe respectively the observed returns $(R_2)_t$, $(R_3)_t$ and $(R_4)_t$, and $(R_1)_t$ while $(R_5)_t$ can be modeled by a simple white noise. In Table 3 we provide the estimated parameters of the AR part with the corresponding standard error. For further analysis

[4] GH distributions can model asymmetry and heavytailedness ant its density is written as:

$$gh(x|\lambda, \alpha, \beta, \delta, \mu) = a(\lambda, \alpha, \beta, \delta, \mu)(\delta^2 + (x-\mu)^2)^{(\lambda-1/2)/2} K_{\lambda-1/2}(\alpha\sqrt{\delta^2 + (x-\mu)^2}) exp((\beta(x-\mu))$$

with $a(\lambda, \alpha, \beta, \delta, \mu) = (\alpha^2 - \beta^2)^{\lambda/2} / (\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2}))$ where K_λ is the modified Bessel function of the third kind, $\delta \geq 0$ and $0 \leq |\beta| < \alpha$. It is symmetric if $\beta = 0$. The normal inverse Gaussian is the special case $\lambda = -1/2$. The hyperbolic distribution is the special case $\lambda = 1$ and many distributions are limiting cases: normal, student-t and variance-gamma - see Prause (1999) or Eberlein and Von Hammerstein (2003).

[5] $AIC = -2ln(L) + 2k$ where k is the number of unknown parameters, n is the sample size and L denotes the maximum likelihood function

AR Par.	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	AIC
Estimates	-0.038	-0.037	-0.068	0.052	-0.077	-0.049	-14677
$(R_1)_t$	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	(0.019)	—
Estimates	-0.0170	0.0461	-0.0588	—	—	—	-17186
$(R_2)_t$	(0.019)	(0.019)	(0.019)	—	—	—	—
Estimates	-0.2435	-0.0372	0.0460	0.0535	—	—	-12784
$(R_3)_t$	(0.019)	(0.020)	(0.020)	(0.019)	—	—	—
Estimates	0.1785	0.0493	0.0634	-0.0796	—	—	-14825
$(R_4)_t$	(0.019)	(0.020)	(0.020)	(0.019)	—	—	—

Table 3: Parameter estimates of AR models for $(R_1)_t$, $(R_2)_t$, $(R_3)_t$ and $(R_4)_t$ with their corresponding standard deviations.

of residuals $\hat{\epsilon_t}$ behavior, volatility models listed previously with skewed Normal, skewed Student and skewed GED have been fitted using QML method. The estimated parameters, standard errors and information criteria (AIC, BIC and HQIC) are given in the sequel (Tables 4 to 8). Note that if more than one model have comparable values of these criteria, then the model with lower number of parameters or lower p-values of significance parameters' tests is chosen. As a result, the best fitting models for each market risk are given hereunder.

- The daily EuroStoxx returns will be modeled using an $AR(6) - EGARCH(2,1)$ process with a skewed student-t distribution.
- The daily 10y-IR returns will be modeled applying an $AR(3) - EGARCH(1,1)$ process with a skewed GED distribution.
- The daily Euro Spread returns will be modeled employing an $AR(4) - EGARCH(1,1)$ process with a skewed GED distribution.
- The daily Euro Implied Volatility returns will be modeled using an $AR(4)-EGARCH(1,2)$ process with a skewed student-t distribution.
- The daily USD/EUR FX returns can be modeled applying an $EGARCH(2,1)$ process with a skewed GED distribution.

These modelings provide the residuals $(\nu_t)_t$ for each market risk that we use in the sequel in order to characterize the dependence structure between these five market risks through the differnet copula approaches.

Models	GARCH Skew Normal	GARCH Skew t	GARCH Skew GED	EGARCH Skew Normal	EGARCH Skew t	EGARCH Skew GED	APARCH Skew Normal	APARCH Skew t	APARCH Skew GED
ω	1.993e-06	1.836e-06	1.986e-06	-0.166	-0.163	-0.168	8.373e-5	9.828e-5	1.081e-4
(σ)	(1.0e-6)	(1.0e-6)	(1.0e-6)	(0.028)	(0.026)	(0.030)	(6.6e-5)	(8.7e-5)	(1.25e-4)
α_1	0.040441	0.022512	0.029267	-0.183506	-0.192990	-0.190	4.765e-2	5.021e-2	5.007e-2
(σ)	(0.022949)	(0.024580)	(0.021716)	(0.034809)	(0.035810)	(0.029)	(0.011)	(0.013)	(0.015)
α_2	0.060573	0.077315	0.071986	0.094335	0.098097	0.098	3.172e-2	2.838e-2	3.084e-2
(σ)	(0.029593)	(0.039019)	(0.031966)	(0.036289)	(0.037286)	(0.030)	(0.014)	(0.016)	(0.016)
β_1	0.892171	0.894842	0.892264	0.980656	0.981269	0.981	9.115e-1	9.150e-1	9.124e-1
(σ)	(0.022899)	(0.024647)	(0.022607)	(0.003199)	(0.002975)	(0.003)	(0.010)	(0.010)	(0.011)
γ_1	-	-	-	0.050233	-0.088864	-0.071	9.999e-1	1	1
(σ)	-	-	-	(0.053973)	(0.047693)	(0.046)	(0)	(0)	(0)
γ_2	-	-	-	0.197870	0.233976	0.218	3.878e-13	3.330e-8	1.447e-9
(σ)	-	-	-	(0.054672)	(0.049043)	(0.046)	(0.310)	(0.436)	(0.412)
δ	-	-	-	-	-	-	1.242	1.195	1.185
(σ)	-	-	-	-	-	-	(0.166)	(0.186)	(0.243)
Skew	0.864093	0.868995	0.877761	0.879882	0.878170	0.883	8.751e-1	8.738e-1	8.792e-1
(σ)	(0.026134)	(0.023062)	(0.024321)	(0.024039)	(0.023254)	(0.023)	(0.023)	(0.024)	(0.023)
Shape	-	10.673368	1.541070	-	12.338142	1.615	-	12.802	1.619
(σ)	-	(2.816239)	(0.093099)	-	(2.843102)	(0.065)	-	(2.890)	(0.065)
AIC	-5.9227	-5.9385	-5.9385	-5.9627	**-5.9736**	*-5.9728*	-5.9571	-5.9666	-5.9667
BIC	-5.9117	-5.9253	-5.9253	-5.9472	**-5.9560**	*-5.9551*	-5.9394	-5.9467	-5.9469
HQIC	-5.9188	-5.9337	-5.9337	-5.9571	**-5.9672**	*-5.9664*	-5.9507	-5.9594	-5.9595

Table 4: Estimates, standard errors and information criteria for various GARCH(2,1), EGARCH(2,1) and APARCH(2,1) models fitted to AR(6) residuals for Euro Stoxx data over the study period.

Models	GARCH Skew Normal	GARCH Skew t	GARCH Skew GED	EGARCH Skew Normal	EGARCH Skew t	EGARCH Skew GED	APARCH Skew Normal	APARCH Skew t	APARCH Skew GED
ω	4.0772e-07	4.0126e-07	4.37526e-07	-0.011699	-0.01963	-0.01902	1.71337e-08	1.4497e-08	1.72832e-08
(σ)	(0.000000)	(0.000000)	(0.000000)	(0.019437)	(0.024215)	(0.029865)	(0.000000)	(0.000000)	(0.000000)
α_1	0.040462	0.036565	0.038961	-0.028143	-0.024134	-0.026605	0.024991	0.023025	0.026732
(σ)	(0.002637)	(0.002827)	(0.002813)	(0.005516)	(0.006465)	(0.006589)	(0.004700)	(0.004342)	(0.006030)
β_1	0.956210	0.959980	0.957081	0.998466	0.997823	0.997931	0.960191	0.963199	0.959153
(σ)	(0.002679)	(0.003030)	(0.003008)	(0.002059)	(0.002547)	(0.003149)	(0.008950)	(0.009348)	(0.010773)
γ_1	-	-	-	0.074884	0.072181	0.074327	0.223489	0.147773	0.196504
(σ)	-	-	-	(0.011301)	(0.013072)	(0.014031)	(0.069774)	(0.067599)	(0.070605)
δ	-	-	-	-	-	-	2.637005	2.660418	2.619477
(σ)	-	-	-	-	-	-	(0.057061)	(0.071057)	(0.078305)
Skew	1.046316	1.045788	1.033857	1.046910	1.046245	1.035340	1.054528	1.049928	1.041894
(σ)	(0.026880)	(0.023203)	(0.024818)	(0.025348)	(0.026977)	(0.024281)	(0.025251)	(0.026739)	(0.024031)
Shape	-	8.928179	1.494864	-	9.414796	1.501410	-	11.332505	1.512204
(σ)	-	(1.401543)	(0.057517)		(1.637031)	(1.646189)		(2.486323)	(0.060645)
AIC	-6.6160	-6.6344	*-6.6359*	-6.6194	-6.6358	**-6.6380**	-6.6176	-6.6344	-6.6372
BIC	-6.6072	-6.6234	*-6.6249*	-6.6084	-6.6226	**-6.6247**	-6.6022	-6.6189	-6.6218
HQIC	-6.6128	-6.6304	*-6.6319*	-6.6154	-6.6310	**-6.6332**	-6.6139	-6.6288	-6.6317

Table 5: Estimates, standard errors and information criterion for various GARCH(1,1), EGARCH(1,1) and APARCH(1,1) models fitted to AR(3) residuals for 10y- IR data over the study period.

Models	GARCH Skew Normal	GARCH Skew t	GARCH Skew GED	EGARCH Skew Normal	EGARCH Skew t	EGARCH Skew GED	APARCH Skew Normal	APARCH Skew t	APARCH Skew GED
ω	9.5974e-06	3.3955e-06	4.35795e-06	-0.246541	-0.084254	-0.134349	5.749273e-05	6.7833e-05	9.9325e-05
(σ)	(0.000000)	(0.000000)	(0.000000)	(0.104423)	(0.062325)	(0.032771)	(0.000107)	(0.000058)	(0.000008)
α_1	7.6106e-02	7.1095e-02	8.1091e-02	0.0462559	0.007297	0.02215	0.087158	0.082246	0.095431
(σ)	(0.005808)	(0.021825)	(0.014820)	(0.025323)	(0.025289)	(0.018344)	(0.030715)	(0.024454)	(0.001407)
β_1	9.0551e-01	9.2791e-01	9.166387e-01	0.96663597	0.988969	0.98283	0.904008	0.93174	0.916515
(σ)	(0.008827)	(0.017732)	(0.010426)	(0.013846)	(0.008150)	(0.004126)	(0.032298)	(0.022195)	(0.001746)
γ_1	-	-	-	0.1781155	0.150668	0.176725	4.6245e-16	6.5271e-14	8.3903e-10
(σ)	-	-	-	(0.041092)	(0.046730)	(0.039230)	(0.103334)	(0.092882)	(0.077800)
δ	-	-	-	-	-	-	1.573837	1.269988	1.276359
(σ)	-	-	-	-	-	-	(0.415692)	(0.164432)	(0.004876)
Skew	1.019547	1.017861	1.033094	1.00392799	1.016939	1.032056	1.017502	1.017661	1.033906
(σ)	(0.034315)	(0.019832)	(0.000844)	(0.031992)	(0.02092)	(0.000831)	(0.033548)	(0.02051)	(0.000409)
Shape	-	4.426968	1.032432	-	4.236312	1.031435	-	4.290887	1.03008
(σ)	-	(0.498421)	(0.060036)	-	(0.455296)	(0.057827)	-	(0.466411)	(0.055613)
AIC	-4.9682	-5.07113	-5.08895	-4.9733	-5.0754	**-5.0925**	-4.9681	-5.0745	*-5.0909*
BIC	-4.9594	-5.060107	-5.07793	-4.962269	-5.062177	**-5.0793**	-4.9549	-5.0591	*-5.0755*
HQIC	-4.9650	-5.06714	-5.084965	-4.9693	-5.07062	**-5.0878**	-4.9681	-5.0689	*-5.0854*

Table 6: Estimates, standard errors and information criterion for various GARCH(1,1), EGARCH(1,1) and APARCH(1,1) models fitted to AR(4) residuals for Euro Spread data over the study period.

Models	GARCH Skew Normal	GARCH Skew t	GARCH Skew GED	EGARCH Skew Normal	EGARCH Skew t	EGARCH Skew GED	APARCH Skew Normal	APARCH Skew t	APARCH Skew GED
ω	6.574e-7	1.048e-6	8.668e-7	-0.101	-0.241	-0.182	3.244e-6	6.297e-6	5.874e-6
(σ)	(2.0e-6)	(1.0e-6)	(1.0e-6)	(3.560e-2)	(6.623e-2)	(0.055)	(1.0e-6)	(1.0e-6)	(1.0e-6)
α_1	;2.526e-1	3.426e-1	3.017e-1	1.212e-3	2.358e-3	2.202e-3	2.671e-1	3.819e-1	3.358e-1
(σ)	(2.664e-2)	(3.673e-2)	(3.554e-2)	(1.117e-3)	(7.013e-3)	(0.004)	(0.022)	(0.036)	(0.032)
β_1	4.450e-1	3.496e-1	4.001e-1	0.657	0.694	0.679	4.482e-1	3.484e-1	3.980e-1
(σ)	(8.402e-2)	(7.983e-2)	(9.084e-2)	(6.369e-2)	(6.417e-2)	(0.068)	(0.076)	(0.075)	(0.078)
β_2	3.015e-1	3.068e-1	2.971e-1	0.331	0.282	0.303	3.091e-1	3.192e-1	3.107e-1
(σ)	(7.099e-2)	(6.653e-2)	(7.530e-2)	(6.384e-2)	(6.456e-2)	(0.069)	(0.072)	(0.067)	(0.070)
γ_1	-	-	-	0.481	6.146e-1	0.558	4.937e-1	3.596e-9	1.663e-7
(σ)	-	-	-	(3.348e-2)	(4.590e-2)	(0.043)	(0.030)	(0.032)	(0.034)
δ	-	-	-	-	-	-	1.694	1.653	1.624
(σ)	-	-					(0.084)	(0.045)	(0.048)
Skew	1.036	1.045	1.047	1.047	1.054	1.060	1.037	1.048	1.051
(σ)	(2.346e-2)	(2.704e-2)	(1.950e-2)	(2.434e-2)	(2.747e-2)	(0.023)	(0.024)	(0.029)	(0.024)
Shape	-	7.653	1.448	-	6.427	1.401	-	6.965	1.426
(σ)	-	(9.320e-1)	(5.675e-2)	-	(0.752)	(0.052)	-	(0.791)	(0.051)
AIC	-7.3240	**-7.3680**	-7.3559	-7.3137	-7.3589	-7.3489	-7.3239	*-7.3680*	-7.3569
BIC	-7.3129	**-7.3536**	-7.3426	-7.3005	-7.3435	-7.3335	-7.3085	*-7.3540*	-7.3392
HQIC	-7.320	**-7.3620**	-7.3511	-7.3089	-7.3533	-7.3433	-7.3183	*-7.3617*	-7.3505

Table 7: Estimates, standard errors and information criterion for various GARCH(1,2), EGARCH(1,2) and APARCH(1,2) models fitted to AR(4) residuals for Euro Implied volatility data over the study period.

Models	GARCH Skew Normal	GARCH Skew t	GARCH Skew GED	EGARCH Skew Normal	EGARCH Skew t	EGARCH Skew GED	APARCH Skew Normal	APARCH Skew t	APARCH Skew GED
ω	7.685e-8	7.743e-8	7.374e-8	-0.030	-0.035	-0.032	7.394e-10	1.187e-8	2.379e-9
(σ)	(1.0e-6)	(1.0e-6)	(1.0e-6)	(0.021)	(0.021)	(0.026)	(1.0e-60)	(1.0e-6)	(1.0e-6)
α_1	2.515e-4	8.549e-7	2.949e-5	0.022	0.006	0.012	1.332e-2	1.230e-2	1.146e-2
(σ)	(3.92e-4)	(0.019)	(4.2e-5)	(0.018)	(0.006)	(0.011)	(0.013)	(0.018)	(0.015)
α_2	2.960e-2	3.133e-2	3.053e-2	-0.021	-0.007	-0.012	2.052e-2	2.233e-2	2.030e-2
(σ)	(2.181e-3)	(0.019)	(1.496e-3)	(0.018)	(0.006)	(0.011)	(0.007)	(0.014)	(0.015)
β_1	9.691e-1	9.677e-1	9.684e-1	0.997	0.997	0.997	9.475e-1	9.353e-1	9.496e-1
(σ)	(2.281e-3)	(0.002)	(1.675e-3)	(0.002)	(0.002)	(0.003)	(0.015)	(0.011)	(0.014)
γ_1	-	-	-	-0.095	-0.120	-0.108	2.367e-2	2.383e-2	2.374e-2
(σ)	-	-	-	(0.044)	(0.046)	(0.054)	(0.091)	(0.070)	(0.011)
γ_2	-	-	-	0.163	0.189	0.177	2.369e-2	2.366e-2	2.371e-2
(σ)	-	-	-	(0.044)	(0.047)	(0.054)	(0.074)	(0.060)	(0.030)
δ	-	-	-	-	-	-	2.760	2.725	2.752
(σ)	-						(0.077)	(0.047)	(0.085)
Skew	9.908e-1	9.908e-1	9.930e-1	0.992	0.992	0.994	9.888e-1	9.984e-1	9.899e-1
(σ)	(2.428e-2)	(0.025)	(2.270e-2)	(0.024)	(0.025)	(0.022)	(0.022)	(0.024)	(0.023)
Shape	-	12.621	1.564	-	11.926	1.555	-	9.983	1.575
(σ)	-	(2.851)	(6.207e-2)	-	(2.698)	(0.065)	-	(1.322)	(0.052)
AIC	-7.3574	-7.3648	**-7.3697**	-7.3531	-7.3611	*-7.3658*	-7.3246	-7.3536	-7.3625
BIC	-7.3464	-7.3516	**-7.3564**	-7.3376	-7.3435	*-7.3482*	-7.3091	-7.3337	-7.3426
HQIC	-7.3534	-7.3601	**-7.3649**	-7.3475	-7.3547	*-7.3595*	-7.3190	-7.3464	-7.3553

Table 8: Estimates, standard errors and information criterion for various GARCH(2,1), EGARCH(2,1) and APARCH(2,1) models fitted to FX USD/EUR returns data over the study period.

4 Dependence Structure between Market Risks

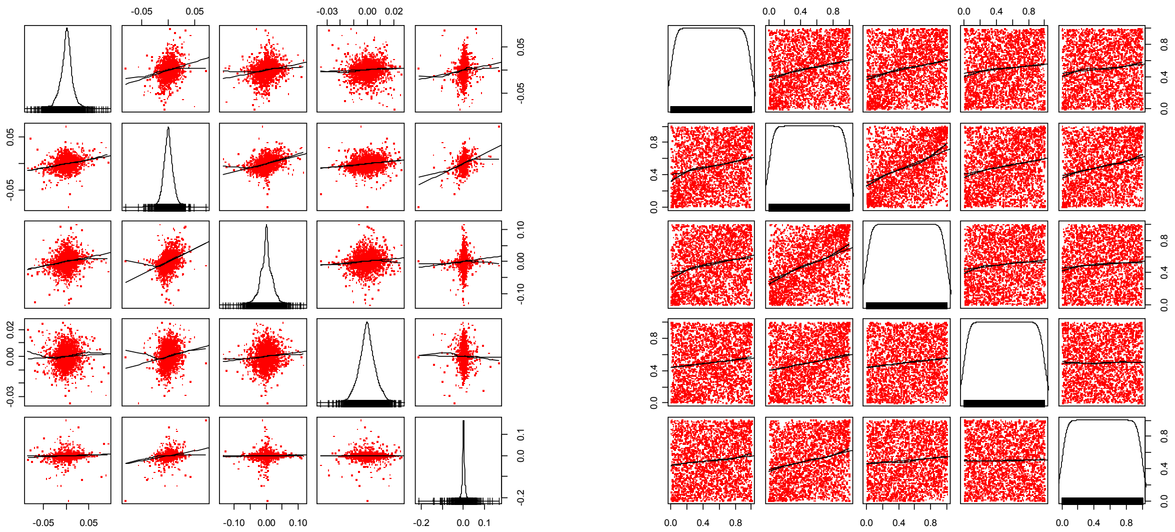

Figure 1: Plots of the filtered data (left) and normalized ranks (right).

In order to account for possible time-dependence and volatility on one hand and to fulfill the assumptions of maximum likelihood used for selecting and calibrating copula models on the other hand, the financial time series for Equity, Interest Rate, Spread, Implied volatility and FX risks have been filtered by applying AR-*like*GARCH models. Given the resulting residuals, respectively denoted $((\nu_1)_t, \dots, (\nu_5)_t)$, the joint density of several copula models are estimated by using a semi-parametric method called Canonical Maximum Likelihood estimation (CML). Indeed, the residuals $(\nu_t)_t$ are transformed via the empirical cumulative distribution function into uniform random variables $U_t = F_N((\nu_t))$ on the unit hypercube. Note that the filtering process do not alter the dependence structure when using copulas.
The resulting "pseudo observations" are used to estimate the unknown parameters of the copula models by using MPLE. Goodness-of-fit measures are the log-likelihood and BIC criterion (to account for the number of parameters). Given their properties, the parametric copulas used in our study are the following:

- Copulas with no tail dependence such as the normal copula,
- Copulas with upper tail dependence only such as the Gumbel copula,
- Copulas with lower tail dependence only such as the Clayton copula,
- Copulas with symmetric upper and lower tail dependence such as the t copula.

In the sequel, we are going to treat two ways of modeling the dependence structure: the variance-covariance matrix approach which is the most currently used approach in the industry and copulas.
Among the copula methodology, we consider 5-variates copulas, FNAC and C-vines. The

maximized log-likelihood function and BIC criterion for each copula model are given in Tables 12, 13 and 16. Note that in order to have a positive dependence overall and an appropriate copula fit, the negative residuals for Spread and Implied Volatility data are taken. Figures 9 and 10 display the plot of the residuals $(\nu_i)_t$ for $i = 1, \dots, 5$ and the plot of the normalized ranks in the unit hypercube.

4.1 Variance-Covariance Matrix Approach

The variance-covariance matrix approach is a convenient and commonly used analytical technique that allows managers to combine marginal distributions of losses or distinct tail losses into a single aggregate loss distribution. The sole requirement is to characterize the level of interdependence of standalone losses, which is typically accomplished with a matrix of linear correlations. The lower the correlations on the non-diagonal elements of the matrix, the greater the level of diversification that can be realized with incremental (long) exposure to a risk component.

The variance-covariance matrix approach is well known from multivariate normal distributions where the value-at-risk (VaR) of a risk category $X := X_1 + \cdots + X_n$ is calculated based on the VaRs of the single risks and their correlation matrix Σ. In order to introduce this approach, we start with the formula for the standard deviation. This formula holds true for all random variables with finite variances. The standard deviation σ of X is calculated as

$$\sigma^2 = (\sigma_1, \dots, \sigma_n)\Sigma(\sigma_1, \dots, \sigma_n)^T,$$

where σ_i denotes the standard deviation of the single risk X_i. When $(X_1, \dots, X_n)$ are multivariate normally distributed and centered, a similar formula can be used for calculating the value-at-risk of X:

$$VaR(\alpha) = \sqrt{\sum_{i=1}^{n} VaR_i^2(\alpha) + 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{i,j} VaR_i(\alpha) VaR_j(\alpha)} \qquad (12)$$

where $\rho_{i,j}$ is the correlation between risks (here the five assets) i and j and α is the confidence level of the aggregated VaR.

More generally, this formula holds true for all centered elliptical distributions. Typical examples of elliptical distributions are the Normal and the Student-t distributions. The latter are quite flexible in what concerns the modeling of heavy tails and the corresponding extreme events. This aggregation approach is based on the assumption that risks are measured as the difference between value-at-risk and the mean.

The main advantages of this approach are that it is simpler relative to other methods, can be evaluated through a simple formula and does not require fundamental information about the lower-level risks. However, while the variance-covariance matrix approach is a simple and highly tractable approach to risk aggregation, the cost to the unwary user is that it effectively fills in unspecified details about the nature of the loss distributions, which may or may not be accurate or intended (e.g. assumption of elliptical distributions). The variance-covariance matrix method imposes thus a simple dependence structure on what is believed to be a more complex web of dependences.

In order to derive the economical capital through VaR and ES (the risk measures we consider in this paper) from the variance-covariance matrix approach, we used an empirical correlation matrix inferred from the historical data. Table 11 displays the correlation matrix between the five market sub-risks. This correlation matrix fulfills the necessary assumptions: symmetry and

positive semi-definiteness.
The overall levels of correlation between market sub-risks are generally in line with what can

	Equity	IR	Spread	FX	IV
Equity	1	0.23	0.21	0.09	0.12
IR	-	1	0.40	0.17	0.25
Spread	-	-	1	0.11	0.08
FX	-	-	-	1	0.03
IV	-	-	-	-	1

Table 9: Correlation matrix for market risks.

be intuitively expected.
Note that the simple summation method is a particular case of the variance-covariance matrix approach when all correlation coefficients are set to 100%. This approach does not allow for diversification and is known to yield usually very conservative economic capital.

4.2 A more general approach: Copulas

4.2.1 5-variate copulas

We first fit 5-variate copulas to the transformed data set of market risks, among the Gaussian, the Student, the Gumbel and the Clayton. We restrict to these copulas since the two first ones are widely used in the industry and because all verify the constraints allowing to perform the calculation. We then estimate the parameters of these copulas and compute the corresponding maximized log-likelihood function and BIC criterion. These values are provided in Table 12. From these findings, the best fitting copula is given by the Student-t with $\nu^* = 9$ degrees of freedom. Note also that, in that case, the elliptic copulas show a better fit than the Archimedean copulas because the former present more parameters to describe the dependence structure than the latter ones.

5-variates copulas	ll.max	BIC
Student-t	**641.566**	**-1196.335**
Gaussian	473.539	-868.172
Gumbel	231.83	-455.787
Clayton	294.471	-581.053

Table 10: Values of the maximized log-likelihood function and BIC criteria for the elliptic and Archimedean 5-variate copulas fitted to the transformed data set.

4.2.2 FNAC models

In a second time, we apply the FNAC models. Once again, the Gumbel and the Clayton copulas are used since these are the only Archimedean copulas that one can employ to model FNAC. Note that in this approach, the dependence structure between our five market risks is described through four parameters instead of only one as in the previous methodology. This really improves the fit of the Archimedean copulas. Regarding the structure, we first arrange the variables according to their dependence (Interest Rate, Spread, Equity, FX and Implied Volatility risks) then in terms of economic capital (Equity, Spread, Implied Volatility, Interest rates and FX). However, both structures do not satisfy the constraint of decreasing dependence. Therefore, this approach will not be used for the purpose of economic capital calculation.

4.2.3 C-Vines

Kendall's tau	Equity	IR	Spread	FX
IV	0.0736	0.1521	0.0545	0.0045
Equity	-	0.1459	0.1425	0.0784
IR	-	-	0.2806	0.1261
Sp	-	-	-	0.0754

Table 11: Sample measures of dependence: Kendall's tau.

Spearman's rho	Equity	IR	Spread	FX
IV	0.1082	0.2224	0.0818	0.0064
Equity	-	0.2127	0.2109	0.1141
IR	-	-	0.4029	0.1833
Sp	-	-	-	0.1121

Table 12: Sample measures of dependence: Spearman's rho.

In order to use the vine methodology, we first compute the samples measures of Kendall's tau and Spearman's rho. Both measures give the same results to determine the C-vine structure we to use to describe the dependence structure between our five market risks. The values of these samples measures are given in Tables 14 and 15. The variable that is the most correlated with the other variables is taken as the main node of the tree. In our case, Interest rate (IR) risk is the key variable of the first tree. Then comes as the main node Spread risk conditionally on IR risk in the second tree, IV risk conditionally on IR and Spread risks in the third tree and finally Equity risk conditionally on IR, Spread and IV risks in the last tree.
In other words, we have in the first layer, the dependence between the variable "IR" and all the other variables in this system is modeled with bivariate copulas. The second layer consists in modeling the dependence of variables *Spread* with variables IV, *Equity* and FX, conditionally on the variable IR. In the last layer, one uses a bivariate copula to model the dependence between the variables *Equity* and FX, conditionally on the variables IR, IV and *Spread*. Finally, we estimate this dependence structure. In Table 16, we provide the values of maximized log-likelihood function and BIC criterion for the different C-Vines copula models.
Based on these figures, it appears that the Student C-vine copula model is the model that best matches our market data set. It is important to keep in mind that during all this exercise, we estimate non-parametrically the margins by using the empirical distributions.
From our findings, one can say that the multivariate Student-t and the Student-t C-vine model appear to be the best approaches of fitting copulas. Then, using pair-copula construction models really improves the fit of the Gumbel and Clayton copulas. Note that the C-vines show a better fit than the 5-variates copula models. Eventually, all the above copula models are used to estimate the economic capital that is needed to cover losses at a certain risk tolerance level through the calculation of the VaR(99.5%) and the ES(99%).

C-vine pair-copulas	ll.max	BIC
Student-t	**646.399**	**-1134.986**
Gaussian	473.539	−868.172
Gumbel	405.357	−731.809
Clayton	459.796	−840.687

Table 13: Values of the maximized log-likelihood function and BIC criteria for the elliptic and Archimedean C-Vine fitted to the transformed data set.

5 Economic capital and Diversification benefits

In this section, the total market economic capital is given through the calculation of the VaR and the ES from the previous copulas modelings. The risk measures are computed based on a specified level of probability. For our purposes, we have chosen $q = 99.5\%$ for the VaR and $q = 99\%$ for the ES. These confidence levels are chosen to be respectively consistent with Solvency II and SST requirements.
In order to achieve our purpose, we have created a virtual portfolio consisting of 61 underlying investments representing the five risks with respective weights $\omega_1, \ldots, \omega_{61}$ so that the change in value of the portfolio over the given holding period (the so-called P&L or profits and losses) can be written as $X = \Sigma_i \omega_i X_i$, where X_i denotes the change in value of the i^{th} investment. Measuring the risk of this portfolio essentially consists in determining its distribution function in order to derive then the functionals describing this distribution function such as its mean, the 99^{th} quantile and the 99.5^{th} quantile.
Knowing that, let us consider this virtual portfolio and a fixed time horizon, and denote by $F_L(l) = P(L \leq l)$ the distribution of the corresponding loss distribution. Given some confidence level $\alpha \in (0, 1)$, the VaR of the portfolio at the confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1 - \alpha)$. Formally,

$$VaR_\alpha = inf(l \in R : P(L > l) \leq 1 - \alpha) = inf(l \in R : F_L(l) \geq \alpha). \tag{13}$$

For an integrable loss L with continuous distribution function F_L and any $\alpha \in (0, 1)$ we have $ES_\alpha = E(L|L \geq VaR_\alpha)$. Note that this risk measure is coherent and reflects the severity of the losses which is not the case for the VaR measure (Artzner et al, 1997).
Now, let denote by μ the mean of the loss distribution. The statistics $VaR_\alpha - \mu$ and $ES_\alpha - \mu$ are used for economic capital purposes instead of ordinary VaR and ES measures.
In our study, the VaR measure is calculated as the point on the ranked P&L distribution that corresponds to the particular level of q. That is, for each output distribution, of the 110000 simulated P&L, the 99.5^{th} VaR corresponds to the 550^{th} worst value if the distribution is ranked in decreasing magnitude. To calculate the ES, we simply take the arithmetic average, or expected value, of the values subsequent to the value corresponding to the VaR measure. Therefore, 99^{th} ES is calculated as the average of the 1^{st} to 1100^{th} worst value of the ranked P&L distribution. Continuing with the procedure outlined earlier, the simulated P&L for each market sub-risk is then aggregated to produce a distribution of the aggregate P&L under each copula model. The single 99.5^{th} VaR and $99^{t}h$ES for the five market sub-risks are displayed in Table 17.
We then conduct benchmarking of these models by comparing the two risk measures and the associated diversification benefits across the different risk aggregation approaches. Actually, we compare the estimated copula models with the variance-covariance matrix approach and the simple summation. We also examine the resulting impact in terms of economic capitals when using these different approaches models. We then present the resulting economic capitals

Market risks	VaR(99.5^{th})	ES(99^{th})
Equity	4840	4918
Interest Rates	872	902
Spread	2341	2453
FX	383	393
Implied Volatility	1897	1916

Table 14: Single VaR(99.5^{th}) and ES(99^{th}) for the five market sub-risks (in EUR m).

Methodologies	$(99.5^{th})VaR$	(99^{th})ES	Diversification Benefits
Variance-covariance approach	6800	6956	34.2%
Simple summation	10334	10582	0%
Gaussian	6063	6220	41.2%
Student	6188	6387	40.0%
Gumbel	6026	6173	41.7%
Clayton	6276	6444	39.2%
C-Vine Student	7303	7502	29.2%
C-Vine Clayton	7467	7674	27.6%
C-Vine Gumbel	7159	7355	30.6%

Table 15: Single VaR(99.5^{th}), ES(99^{th}) (in EUR m) and diversification benefits (in %) for both linear and copula approaches.

hereafter in monetary units. To make the assessment, Table 18 exhibits the economic capitals for each linear approach and for each pair-copula model and their corresponding diversification benefits in percentage.

5.1 Concluding Remarks

From Table 18, we observe that the simple summation approach is the most conservative method since it does not allow for diversification, which is equivalent to say that the risks are perfectly positively correlated. Then comes the C-vine copula approach followed then by the variance-covariance matrix method. We also note that the use of 5-variates copulas leads to lower aggregated economic capital than when using the variance-covariance matrix approach.

When focusing on the copula methodology, we see that there is a consistent effect due to the choice of the structure on the total economic capital regardless of the risk measure used. The most conservative copula approach is the C-vine approach. This can be explained by the fact that we have a more appropriate structure for gathering the variables and more parameters to describe the dependence pattern than when applying the other copula approach.

The choice of copula has also a profound effect on the resulting economic capital and the diversification benefits. As we expected, due to its lack of account of the lower tail dependence of losses, the Gumbel copula results in the lowest capital requirement while the Clayton and the Student copulas result in the highest capital requirement. From these findings, we can conclude that among the copula approaches, the Clayton C-vine copula results in the highest capital requirement (which makes sens because it allows for lower tail dependence), followed by the Student C-vine. This difference is significant for both risk measures with the extreme being the

ES (99%) case. We can also underline that the observed discrepancy of the economic capital between copulas highlights the importance of correctly modeling the dependence structure, in particular the structure and tail dependence between losses across risks.
We make the following key observations:
In Table 18, the value of the diversification benefits according under each copula is calculated and this provides an initial crude measure of the magnitude of the diversification benefits resulting from each copula. In decreasing order, the ranking of the copulas is 5-variates Gumbel (41.7%), 5-variates Gaussian (41.2%) and 5-variates Student (39.6%). Therefore, modeling dependencies between risks from different risks with 5-variates copulas allowing for higher tail dependence will give by far the highest level of diversification benefits while using a C-vine that allows for lower tail dependence will result in the lowest level of diversification benefits.
Then, the choice of the structure and the choice of copula have a paramount effect on the economic capital as well as on the diversification benefits for an insurer. This effect is driven mainly by the amount of tail dependence that the copula allows for losses between risks. The more lower tail dependence allowed by a copula, the higher the economic capital is if losses are aggregated under that copula. Hence, even if the best fitting copula models are the Student-t C-vine and the 5-variate Student-t copula, the copulas that exhibit lower tail dependence will yield the highest EC figures. This result makes sense due to the nature of these copulas.
For all copula assumptions except the C-vine model, we see that they underestimate the economic capital for the virtual portfolio. This means that if insurers are willing to use a 5-variates copula based model rather than a vine method to aggregate their risks for economic capital purposes, they may not cover their risks to withstand unexpected losses. Therefore, in making that assertion, we must be aware that this result is sensitive to the choice of the copula.
Consequently this leads to the need for more accurate and flexible internal models to be developed for capital determination purposes where copulas are incorporated for the aggregation of losses from different risks. However, as the actual economic capital are found to be extremely sensitive to the choice of copulas, and further this sensitivity varies depending on the portfolio composition of the particular insurer, it is imperative that in constructing the internal models, an appropriate copula assumption and an appropriate structure are made.

6 Conclusion

To ensure solvency, insurers are required both for regulatory purposes and as a going business concern to hold capital to back their insurance liabilities. In aggregating losses from different risks for the purpose of capital determination, insurers have traditionally either ignored the dependence structure between risks or used simple linear correlations to model such dependence. In this paper, we aggregate each risk losses using the variance-covariance matrix approach, simple summation, 5-variates copulas and two variants of pair-copula models. We then assess the capital requirements in each case using the value-at-risk and tail conditional expectation risk measures at 99.5% and at 99%, respectively. Further, we calculate the diversification benefits from holding capital in aggregate risks. We analyze this diversification benefit for the different copula assumptions. The following are the key findings of this paper.
First, the choice of the structure for gathering variables has a huge effect on both the economic capital and diversification benefit for an insurer. In our case study, the more sophisticated or appropriate structure a model allows (vines), the higher the required capital is. The opposite relationship between the choice of the structure and the diversification benefit exists. Therefore, because of the potential for massive modeling errors, it is imperative for insurers to select a structure that is most reflective of their own unique situation to avoid the risk of mis-calculating their capital requirement.

Second, the choice of the copula model impacts also both the economic capital and diversification benefits for an insurer. Indeed, the more tail dependence a copula allows (here Clayton and Student), the higher is the required capital.
Lastly, the adequacy of the variance-covariance matrix approach is assessed against the capital requirements implied by the copula models.The C-vine Student and the C-vine Clayton appear to yield more conservative economic capital than when using the variance-covariance matrix approach. Again, the higher the copula's allowance for tail dependence and the more appropriate structure a copula model allows, the higher the resulting capital requirement will be under that model.
The reader of this paper should be aware that there are simplifying assumptions inherent in our analysis. In calculating the capital requirements and diversification benefits, we have limited our focus on the five main market sub-risks of an insurer while ignoring other sources of market risk such as for instance hedge funds, private equity and real estate risks. However, mainly due to data limitations, amongst other reasons, we have excluded the former risks from our scope. Due to these assumptions, the single most significant limitation to the results of this paper is that they merely serve to compare the effect of the dependence structure on the capital requirements and does not quantify the required capital that a particular insurer should hold for the total market risk. The numerical results in monetary terms presented in this paper must also be taken with caution as this is only representative of a virtual portfolio held by an insurer and again, we emphasize that it only accounts for the market risk component of the total capital requirements. However, the modeling procedure of aggregating risks using copulas as demonstrated in this paper can be readily adapted.
Given that our findings in this paper indicate a pronounced effect of the copula structure on the capital requirements for insurers, we suggest further research into using other copulas to model the dependence structure of insurers portfolios. In this paper, we have explored using pair-copulas to model the dependence structure but these are only a few of the vast pool of copula structures that one can draw from. For example, rather than modeling in a static way the dependence between risk sources, one may investigate dynamic dependencies between these risks using dynamic pair-copula models. It would be also interesting to conduce the same analysis considering parametric margins instead of empirical ones. All these topics will be discussed in future papers.

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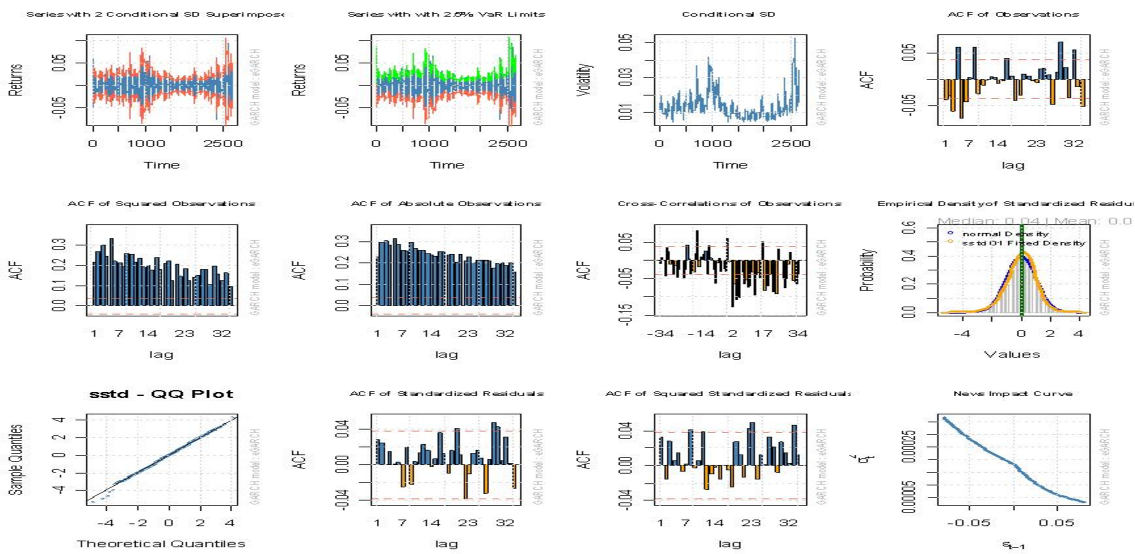

Figure 2: Diagnostic plots of standardized residuals from the fitted AR6-EGARCH(2,1) model with skewed student distribution function for Equity risk data

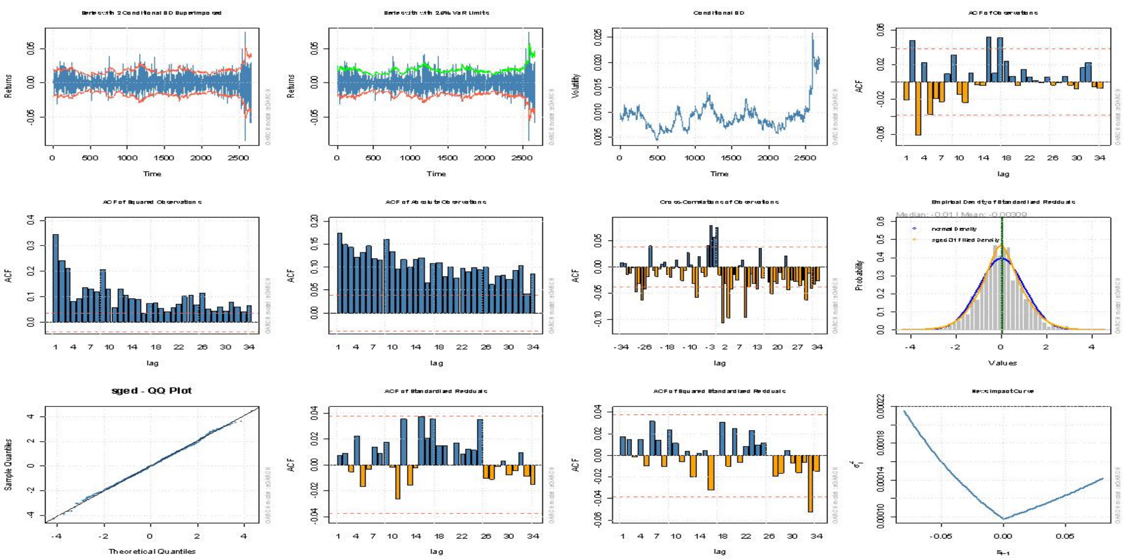

Figure 3: Diagnostic plots of standardized residuals from the fitted AR3-EGARCH(1,1) model with skewed GED distribution function for IR risk data

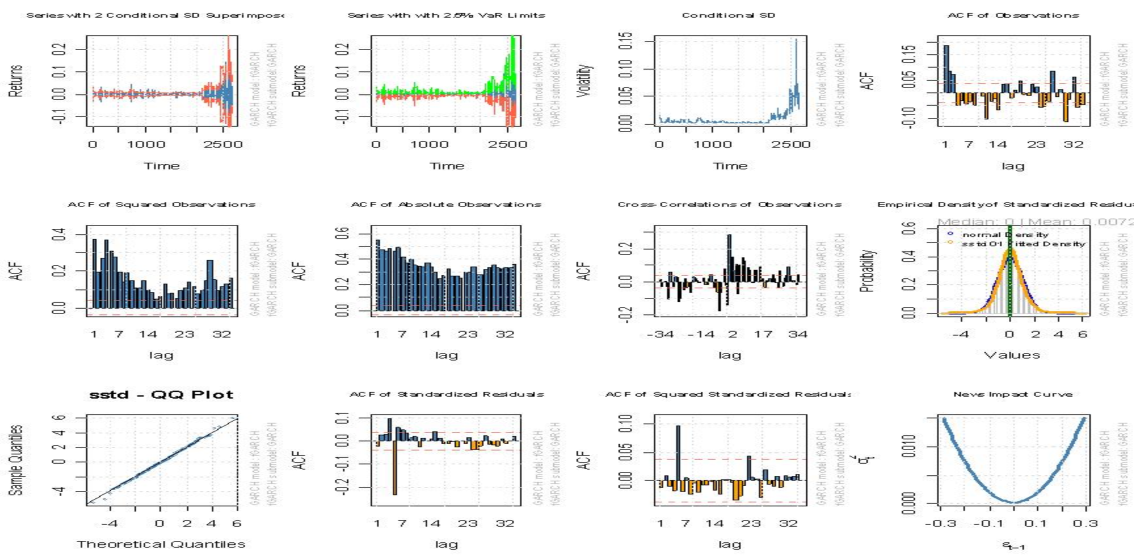

Figure 4: Diagnostic plots of standardized residuals from the fitted AR4-GARCH(1,2) model with skewed Student distribution for Implied Volatility risk data

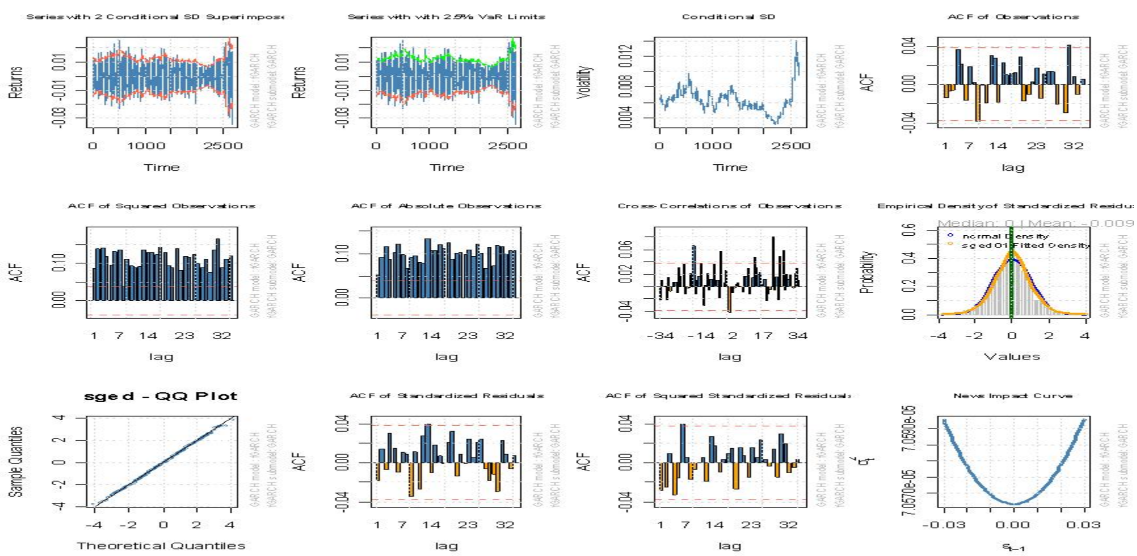

Figure 5: Diagnostic plots of standardized residuals from the fitted GARCH(2,1) model with skewed GED distribution for FX risk data

Parameters	μ	λ	σ	γ	α	K-S test (p-value)
$(R_1)_t$	4.595e-11	8.025e-1	1.554e-2	0	-	0.110 (0.197)
$(R_2)_t$	5.183e-11	0.5	9.501e-3	0	-	0.121 (0.117)
$(R_3)_t$	5.902e-13	-7.485e-2	2.969e-1	0	1.772e-12	0.114 (0.192)
$(R_4)_t$	2.026e-11	0.5	8.003e-3	0	-	0.122 (0.112)
$(R_5)_t$	-6.190e-5	1.903	6.511e-3	0	-	0.124 (0.103)

Figure 6: Estimated parameters and Kolmogorov-Smirnov statistics of VG distribution for $(R_1)_t$, $(R_2)_t$, $(R_4)_t$ and $(R_5)_t$ and of GHYP distribution for $(R_3)_t$.